

On quantum mechanics in Friedmann–Robertson–Walker universe

E A Tagirov

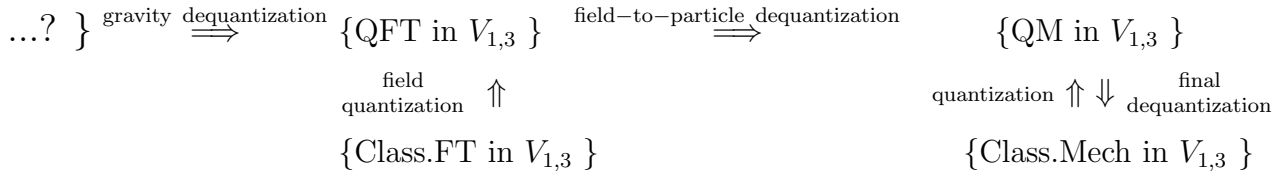
N N Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna, 141980, Russia,
e-mail: tagirov@thsun1.jinr.dubna.su

Abstract

It is shown that only in the space-times admitting a 1+3-foliation by flat hypersurfaces (i.e., in the Bianchi I type space-times, the isotropic version of which is the spatially flat Friedmann–Robertson–Walker space-times) the canonical quantization of geodesic motion and the quantum mechanical asymptotics of the quantum theory of scalar field lead to the same canonical commutation relations (CCR). Otherwise, the field-theoretical approach leads to a deformation of CCR (particularly, operators of coordinates do not commute), and the Principle of Correspondence is broken in a sense. Thus, the spatially flat cosmology is distinguished intrinsically in the quantum theory.

In my paper [1], the generally covariant quantum mechanics (QM) of a neutral spinless point-like particle in *the general Riemannian space-time* $V_{1,3}$ was formulated as a one-particle approximation in the quantum theory of scalar field (QFT) φ . This field-theoretical (FT) approach QM in $V_{1,3}$ is a natural alternative to approaches based on quantization of the corresponding classical mechanics, that is the geodesic dynamics. There are different procedures of quantization: canonical (see for application in $V_{1,3}$, e.g., in [2, 3] and references therein), paths integration (see [2, 7]), quasiclassical and geometrical (see, e.g., [4, 5, 6]). However, it should be noted that, except [3, 6], the mentioned as well as numerous other papers devoted to the "free motion" on curved background consider only the case of $V_{1,3} \sim T \otimes V_3$ for which the geodesic lines in $V_{1,3}$ and V_3 are in the one-to-one correspondence. For a time-dependent $V_{1,3}$, this restriction is equivalent to the non-relativistic approximation.

A natural quantum hierarchy of the fundamental physical theories can be represented by the following diagram (I use the term 'dequantization' to denote a transition which is opposite to quantization):



Do the procedures along the arrows starting from the classical field theory and classical mechanics lead to the same QM, at least, in the simplest case of spinless and chargeless particle and field? They do almost trivially in the case of the Minkowski space-time $E_{1,3}$ if the Cartesian coordinates are used. However, the situation changes drastically in the case of $V_{1,3}$: the 'field-to-particle dequantization' and 'quantization' arrows on the diagram lead generally to different QM in $V_{1,3}$ even in the case of $T \otimes V_3$. Therefore, the divergence is not caused by the particle creation and annihilation processes. The unique class of $V_{1,3}$, when one can make consistent the results of the two approaches to QM is the Bianchi I type of which the spatially flat Friedmann–Robertson–Walker (FRW) space-time is the isotropic subclass. This is the main assertion in this my letter.

I should like to think it is not an accidental coincidence that the spatially flat FRW space-time is apparently realized apparently in our Universe. At least, astrophysical data testify more and more convincingly, see, e.g., [8], to that

$$\Omega \stackrel{def}{=} \frac{\rho_b + \rho_d + \rho_\Lambda}{\rho_c} = 1$$

where ρ_b is the average density of the ordinary (barionic) matter in the Universe ρ_d is the density of the dark matter, ρ_c is the critical density corresponding to the boundary between the spheric and hyperbolic geometries of the space and $\rho_\Lambda = \Lambda/(3H_0)^2$ is the effective contribution of the cosmological constant Λ , H_0 being the Hubble constant. Of course, there should be a very deep reason for that the spatial curvature in the Universe is equal or very close to zero, the

value which is critical between the continua of possible closed and open FRW cosmologies. One might say joking that there exists some sort of "quantropic principle" which fixes the geometry so that internally consistent quantum theory could exist.

Consider the situation with QM in $V_{1,3}$ and especially in the FRW space-times in some detail on the basis of [1, 3].

As concerns traditional operator quantization of a Hamiltonian system, the canonical commutation relations (CCR)

$$[\hat{p}_{(i)}, \hat{p}_{(j)}] = 0, \quad [\hat{q}^{(i)}, \hat{q}^{(j)}] = 0, \quad [\hat{q}^{(i)}, \hat{p}_{(j)}] = i\hbar\delta^{(i)}_{(j)} \cdot \hat{1} \quad (1)$$

take a central place among its postulates. Here, the basic operators $\hat{q}^{(i)}$ and $\hat{p}_{(j)}$, $i, j, k, \dots = 1, 2, 3$, of position and conjugate momentum observables acting on a Hilbert space of states \mathcal{H} correspond to the Darboux coordinates $q^{(i)}, p_{(j)}$ on the phase space $T^*\Sigma_3$, the cotangent bundle over a Cauchy hypersurface Σ_3 . (Closing indices in the parentheses denotes that the former refer to the phase space.) Thus, Σ_3 is the configurational space and provides $V_{1,3}$ by a 1+3-foliation (a frame of reference in the physical terms) by a one-parametric system of Cauchy hypersurfaces $\Sigma_3\{s\}$ which are normal geodesic translation of $\Sigma_3 \equiv \Sigma_3\{0\}$. The latter is assumed here to be a topologically elementary manifold, which means, in fact, that only local physical manifestations of curvature are taken into account in the region where the normal geodesic congruence has no focal points.

A quantum-mechanical operator \hat{f} corresponding to a given function (classical observable) $f(q, p)$ is supposed to be constructed by some procedure from the basic operators $\hat{q}^{(i)}, \hat{p}_{(j)}$ and the function itself $f(q, p)$.¹ There are infinitely many procedures like that. For our case of motion of a structureless particle, these procedures are equivalent still $V_{1,3} \sim E_{1,3}$, $\Sigma_3 \sim R_3$ and Cartesian coordinates X^i on Σ_3 and momenta P_i conjugate to them are taken. If even one of these three conditions is broken the ambiguity of the quantization map $f(q, p) \rightarrow \hat{f}$ becomes physically essential and manifests itself in some way. In canonical quantization, this problem is known as the problem of ordering of operator products. I leave this very important topic for a special discussion elsewhere; some idea of the problem in canonical quantization and paths integration approaches can be found in [2, 3].

The problem of product ordering in canonical quantization is combined with another ambiguity, namely, a dependence of \hat{f} on choice of coordinates $q^{(i)}$ even a rule for each fixed ordering. This means, for example, that the quantum Hamilton operator determining the dynamics of the system under consideration depends on the choice of classical observables $q^{(i)}, q^{(j)}$. Thus, along the 'quantization' arrow in the diagram above, one comes to infinitely many QMs for the same classical mechanics. Again, in $\Sigma_3 \sim R_3$, there exists a preferred class $q^{(i)} = X^i$ determined by the isometry group of space translations.

¹It should be remarked here that, from the pragmatic physicist point of view of a pragmatic physicist, to describe the quantum motion of a particle, one needs actually only the Hamilton operator in addition to $\hat{q}^{(i)}$ and $\hat{p}_{(i)}$. An alternative FT approach considered below solves just this restricted problem.

I concentrated here on the problems of the canonical operator approach to quantization, but it is known that such popular alternative as paths integration has an equivalent ambiguities, see, e.g., [2]. Restrict now our consideration of QM in $V_{1,3}$ to the general FRW space-time. There exists a natural 1+3-foliation which reduces the metric to the form, see, e.g., [9], Sec.14.2,

$$ds^2 = c^2 dt^2 - b^2(t) \omega_{ij}(\xi; k) d\xi^i d\xi^j, \quad (2)$$

where $\omega_{ij}(\xi; k)$ is the metric tensor of a space section $t = \text{const}$ which is a 3-sphere, a 3-plane and a 3-hypersphere respectively for $k = 1$, $k = 0$ and $k = -1$. The classical Hamilton function $H(q, p; t)$, $\{q, p\} \in T^*\Sigma_3\{t\}$ for a geodesic motion $\xi^i = \xi^i(t)$ is, see, e.g. [3]

$$H(\xi, p; t) = mc^2 \left(1 - \frac{\omega^{ij}(\xi) p_i p_j}{b^2(t) m^2 c^2} \right)^{1/2} \quad (3)$$

We see that the coordinates ξ^i have a two-fold purpose in the Hamilton formalism:

a) to provide $\Sigma_3 \equiv \Sigma_3\{t\}$ by a manifold structure (arithmetization):

$$\Sigma_3 \supset U \xrightarrow{\xi^i} \mathbb{R}_3;$$

b) to be a classical observable of position of the particle as if it were $q^{(i)} \equiv \xi^i$.

It is useful, though not necessary, to separate these purposes using $\{\xi^i\}$ only for purpose a) (that is to consider them as an ordinary coordinate system on Σ_3) and introducing general Darboux coordinates $\{q^{(i)}, p_{(j)}\}$ in the phase space by the following map:

$$\xi^i \longrightarrow q^{(i)}(\xi), \quad p_j \longrightarrow p_{(j)} = K_{(j)}^l(\xi) p_l, \quad K_{(j)}^l \partial_l q^{(i)} = \delta_{(j)}^{(i)}. \quad (4)$$

Thus, the observables of spatial position and momentum are defined by values of $q^{(i)}(\xi)$ and $p_{(i)}(\xi)$ which are scalar fields with respect to diffeomorphisms of $\Sigma_3\{t\}$ for each fixed value of t . Now we can rewrite the Hamilton function as

$$H(q^{(i)}, p_{(j)}; t) = mc^2 \left(1 - \frac{\omega^{(kl)}(q) p_{(k)} p_{(l)}}{b^2(t) m^2 c^2} \right)^{1/2} \quad (5)$$

where $\omega^{(kl)}(q) = \partial_i q^{(k)} \omega^{ij}(\xi) \partial_j q^{(l)}$ is now a scalar with respect to the diffeomorphisms of Σ_3 and a classical observable as a function on the phase space.

Then, the corresponding basic operators can be represented as differential operators acting in $\mathcal{H} = L^2(\Sigma_3; \mathbb{C}; b^3(t) \sqrt{\omega} d^3 \xi)$, (that is, in the space of complex functions $\psi(\xi; t)$ which are square integrable over $\Sigma_3(t)$ with the natural measure $d\sigma \stackrel{\text{def}}{=} b^3(t) \sqrt{\omega} d^3 \xi$ and have the standard Born probabilistic interpretation):

$$\hat{q}^{(i)}(\xi) \stackrel{\text{def}}{=} q^{(i)}(\xi) \cdot \hat{\mathbf{1}}, \quad \hat{p}_{(j)}(\xi) \stackrel{\text{def}}{=} -i\hbar \left(K_{(j)}^l(\xi) \tilde{\nabla}_l + \frac{1}{2} \tilde{\nabla}_l K_{(j)}^l(\xi) \right), \quad (6)$$

where $\tilde{\nabla}_l$ is the covariant derivative determined by the metric tensor ω_{ij} . Hence and throughout the "hat" over characters denotes differential operators with variable coefficients, which act in

$L^2(\Sigma_3(t); \mathbb{C}; b^3(t)\sqrt{\omega} d^3\xi)$ and contain only derivatives along $\Sigma_3(t)$ for fixed t . The Hamilton operator corresponding to $H(q, p; t)$, eq.(3) is obtained from the latter by substitution the operators $\hat{q}^{(i)}(\xi)$, $\hat{p}_{(j)}(\xi)$ and $\hat{\omega}^{(ij)}(\xi) = \omega^{(ij)}(\hat{q}) = \omega^{(ij)}(\xi) \cdot \mathbf{1}$ in some order instead of $q^{(i)}$, $p_{(j)}$ and $\omega^{ij}(\xi)$ into $H(q, p; t)$ Then for any ordering one obtains

$$H_0 \stackrel{\text{def}}{=} \frac{1}{2m} \omega^{(ij)} p_{(i)} p_{(j)} \xrightarrow{\text{quantization}} \hat{H}_0 = -\frac{\hbar^2}{2m} \Delta_\Sigma + V_q(\partial q, \tilde{\nabla} \partial q, \dots; t), \quad (7)$$

and, after a unitary transformation, one has [3]

$$\hat{H}(\hat{q}, \hat{p}; t) = mc^2 \left(\sqrt{1 + \frac{2\hat{H}_0}{mc^2}} - 1 \right), \quad (8)$$

We see that owing to V_q the Hamilton operator and, consequently, the quantum dynamics depends on the choice of observables $q^i(x)$! Thus, even if we postulate a concrete rule of ordering (e.g., Weyl's one is very popular), we nevertheless have an infinite variety of QMs instead of a single firmly established theory. However, in the spatially flat FRW space-time, a preferred choice of $q^{(i)}$ exists: $q^{(i)} = X^i$. If it is done, then not only the quantum potential is fixed as equal to zero, but also the problem of ordering disappears since $\omega^{(ij)} = \text{const}$.

Now, let us go along to the 'field-to-particle' arrow on our diagram above following paper [1]. In this approach, the one-quasiparticle subspace Φ^- of Fock space \mathcal{F} for the free quantum scalar field $\tilde{\varphi}$ is constructed by an analogy with the standard quantum-mechanical concept of localized particle, see, e.g., [10] and a generalization of the concept to $V_{1,3}$ in [3]. This means that

- a) Φ^- is mapped asymptotically in c^{-2} onto $L^2(\Sigma; \mathbb{C}; d\sigma)$, see Sec 4. in [1]; thus, $\psi(x) \in L^2(\Sigma; \mathbb{C}; d\sigma)$ can be considered as the probability amplitude to find the particle at the point $x \in \Sigma$ and further results are comparable with those of the canonical quantization;
- b) operators of basic one-particle observables acting in this $L^2(\Sigma; \mathbb{C}; d\sigma)$ as differential operators are generated by the corresponding field-theoretical (FT) operators acting in \mathcal{F} ;
- c) the Hamilton operator $H(x)$ is determined by asymptotic transformation of the field equation for φ to the Schrödinger equation. The one-quasiparticle subspace Φ^- thus determined may be considered as one-particle subspace of \mathcal{F} .

The field equation is the well-known generalization of the Klein–Gordon–Fock equation to $V_{1,3}$ in the general non-minimal form

$$\begin{aligned} \square \varphi + \zeta R(x) \varphi + \left(\frac{mc}{\hbar} \right)^2 \varphi &= 0, \quad x \in V_{1,3} \\ \square &\stackrel{\text{def}}{=} g^{\alpha\beta} \nabla_\alpha \nabla_\beta, \quad \alpha, \beta, \dots = 0, 1, 2, 3, \end{aligned} \quad (9)$$

$R(x)$ is the scalar curvature of $V_{1,3}$ ($R(x) = R(t)$ in the FRW space-time.), and ζ is a free parameter. This equation generates asymptotically in c^{-2} the following Schrödinger equation

²It is remarkable that now the metric tensor has become, in a definite sense, quantized!

³Below, unlike [1], the operators acting in \mathcal{F} will be denoted by as \tilde{O} to be distinguished from the QM-operators denoted as \hat{O} .

in the FRW space-time for $\psi(x) \equiv \psi(\xi; t) \in L^2(\Sigma_3(t); \mathbb{C}; b^3(t)\sqrt{\omega}(t)d^3\xi)$:

$$i\hbar\mathcal{T}\psi = \hat{H}_N(\xi; t)\psi; \quad \mathcal{T} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \frac{3}{2}\frac{\partial b(t)}{\partial t}. \quad (10)$$

Here \hat{H}_N is the Hamilton operator which is an asymptotic expansion starting with \hat{H}_0 , eq.(7), in which it is taken $V_q = -(\hbar^2/2m)\zeta R$. For $N \rightarrow \infty$, this expansion can formally be partially summed and represented through \hat{H} , eq.(8), as follows:

$$\hat{H}_\infty = \hat{H} + \sum_{n=1}^{\infty} \frac{\hat{h}_n(\xi; t)}{(2mc^2)^n} \quad (11)$$

and the operators $\hat{h}_n(\xi; t)$ are such that they vanish if $[\mathcal{T}, H_0] = 0$. Note that \mathcal{T} is not an operator in $L^2(\Sigma_3(t); \mathbb{C}; b^3(t)\sqrt{\omega}d^3\xi)$. Contrary to the canonical approach, the Hamilton operator is a scalar with respect to diffeomorphisms of $\Sigma_3(t)$ and depends only on choice of the parameter ζ . The latter, in turn, does not affect the main conclusions of the present letter.

The operators of basic observables of position and momentum are now determined asymptotically by some QFT-operators (operators acting in \mathcal{F}). As concerns the operator $\hat{p}_K(x)$ of projection of momentum $\hat{p}_K(x)$ on any given vector field $K^\alpha(x)$, not necessarily directed along $\Sigma_3(t)$, it is natural to determine it through the corresponding QFT-operator for the quantized scalar field $\check{\varphi}(x)$ and given Σ :

$$\check{\mathcal{P}}_K\{\check{\varphi}; \Sigma\} = : \int_{\Sigma} d\sigma^\alpha K^\beta T_{\alpha\beta}(\check{\varphi}) :, \quad (12)$$

where the colons denote the normal product of the creation and annihilation operators in \mathcal{F} and $T_{\alpha\beta}$ is the metric energy-momentum tensor for $\varphi(x)$ [11]; in the FRW case, it is natural to put $\Sigma \sim \Sigma_3(t)$. Then, for $K_{(i)}^\alpha(x) \stackrel{\text{def}}{=} K_{(i)}^j \delta_j^\alpha$ the matrix elements of $\check{\mathcal{P}}_{K_{(i)}}\{\check{\varphi}; \Sigma_3(t)\}$ in Φ^- , being expressed as matrix elements of an operator in $L^2(\Sigma_3(t); \mathbb{C}; b^3(t)\sqrt{\omega}d^3\xi)$, give the field-theoretically determined quantum-mechanical operator of projection momentum of a localizable configuration of the quantum field $\check{\varphi}(x)$. It can be calculated as asymptotic expansion $\hat{p}_{(i),N}(\xi; t)$ from the general and generally covariant expression (53) in [1]. Again, $\hat{p}_{(i),0}(\xi; t) = \hat{p}_{(i)}(\xi; t)$ being defined in eq.(6). In Section 5 of [1] it is shown also that the triple $\hat{p}_{(i),N}(\xi; t)$ is commutative for any N if $K_{(i)}^j$ are commutative Killing vector fields on $\Sigma_3(t)$. *Just in this case $V_{1,3}$ is of the Bianchi I type.* Otherwise, the asymptotic terms deform the first relation in CCR, eq.(1).

It remains now to consider, in the same way, relations involving spatial position observables. Operators of spatial position of a particle in QFT have been considered in [12], [13], but only in terms of the Cartesian coordinates in the Minkowski space-time. However, since the QFT-prototype $\check{\mathcal{P}}_{K_{(i)}}\{\check{\varphi}; \Sigma\}$ for $p_{(i)}(\xi; t)$ exists one may expect that there also exist an analogous generally covariant QFT-operators $\check{Q}^{(i)}$. By analogy with $\check{\mathcal{P}}_{K_{(i)}}$, they should be integrals over Σ of a sesquilinear form of $\check{\varphi}$ and linear functionals of arbitrary scalar functions $q^{(i)}(x)$ which satisfy the conditions $\partial^\alpha \Sigma(x) \partial_\alpha q^{(i)}(x) = 0$, $\text{rank}||\partial_\alpha q^{(i)}(x)|| = 3$ and thus determine

a point on each $\Sigma_3(t)$. It is natural, in the FRW case, to adjust their values with $q^{(i)}(\xi)$ introduced in the canonical approach.

It appears that for a given triple $q^{(i)}(x)$ there is a unique triple of QFT-operators

$$\check{\mathcal{Q}}^{(i)}\{\check{\varphi}; \Sigma_3(t)\} = \int_{\Sigma} d^3\xi b^3(t)\sqrt{\omega} q^{(i)}(\xi) \check{N}(\xi; t) \quad (13)$$

where $\check{N}(\xi; t)$ is the QFT-operator of quasiparticle density. (Here I rewrite the generally covariant eq.(19) from [1] for the particular case of the FRW metric (2).

In the same way, as for $\check{\mathcal{P}}_K\{\check{\varphi}; \Sigma\}$, one comes to the general formula (59) in [1] which is in fact an asymptotic expansion of the form

$$\hat{q}_N^{(i)}(\xi; t) = \hat{q}^{(i)}(\xi; t) + \sum_{n=2}^N \frac{\hat{q}_n^{(i)}(\xi; t)}{(2mc^2)^n} + O(c^{-2(N+1)}). \quad (14)$$

Note that the corrections to $\hat{q}^{(i)}$ start with a term of order $O(c^{-4})$.

It is easy to see from consideration of $\hat{q}_2^{(i)}(\xi; t)$, that operators $\hat{q}_2^{(i)}(\xi; t)$ do not commute except the case when $q^{(i)}(\xi; t) \equiv \xi^i \equiv X^i$, X^i being Cartesian coordinates in a Bianchi I space-time. It remains to show that asymptotic operators \hat{X}_N^i mutually commute for $N \rightarrow \infty$ too. It is not trivial because, owing to a time dependence of the metric, eq.(59) in [1] and its expression in the form (14) give an infinite series for \hat{X}_∞^i . I restrict the consideration by the FRW case; generalization to the Bianchi I type is straightforward. Then, the following proposition can be easily proved.

Proposition.

In the spatially flat FRW space-time, a unitary operator \hat{U} exists such that

$$\hat{U} \hat{X}_\infty^i, \hat{U}^\dagger = X^i \cdot \hat{\mathbf{1}}. \quad (15)$$

Proof. At first, owing to the translational invariance \hat{U} and \hat{X}^i have correspondingly the forms

$$\hat{U} = \sum_{n=0}^{\infty} u_n(t) \Delta^n, \quad \hat{X}_\infty^i = X^i \cdot \hat{\mathbf{1}} + \sum_{n=0}^{\infty} x_n(t) \Delta^n \frac{\partial}{\partial X_i} \quad (16)$$

where Δ is the Euclidean Laplace operator. Therefore eq.(16) can be represented in the form

$$[\hat{U}, X^i \hat{\mathbf{1}}] = \hat{U} \mathcal{X}(t, \Delta) \frac{\partial}{\partial X_i} \quad (17)$$

which is equivalent to the equation

$$\frac{\partial}{\partial s} U(t, s) = \frac{i}{2} \sum_{n=0}^{\infty} x_n(t) s^n U(t, s); \quad U(t, s) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} u_n(t) s^n$$

which determines the coefficients $u_n(t)$ and always has a solution such that the condition of unitarity $\hat{U} \hat{U}^\dagger = \hat{\mathbf{1}}$ is fulfilled. Q.E.D.

Thus, the algebraic structures of basic observables of spatial position and momentum in the canonical and field-theoretical approaches to QM in $V_{1,3}$ are in accordance only when $V_{1,3}$ is of the Bianchi I type and the Cartesian coordinates are taken as the classical position observables. Has this fact any concern to the observed spatial flatness of the Universe, or this is only an accidental coincidence, seems to be a question of fundamental interest. To study consequences of non-commutativity of coordinates seem not less interesting

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